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On the joint residence time of *N* independent two-dimensional Brownian motions

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Abstract
We study the behaviour of several joint residence times of N independent Brownian particles in a disc of radius R in two dimensions. We consider: (i) the time $T_N(t)$ spent by all N particles simultaneously in the disc within the time interval $[0, t]$, (ii) the time $T_N^{(m)}(t)$ which *at least* m out of N particles spend together in the disc within the time interval $[0, t]$, and (iii) the time $\tilde{T}_N^{(m)}(t)$ which *exactly* m out of N particles spend together in the disc within the time interval $[0, t]$. We obtain very simple exact expressions for the expectations of these three residence times in the limit $t \to \infty$.

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1. Introduction

One of the important and physically meaningful characteristics of *individual* Brownian particle trajectories is the time $T(t)$ spent in a finite domain $\mathcal{S}$ within some observation time t. This time is referred to in the literature as the 'occupation' [1, 2] or the 'residence time' [3–7]. Note that contrary to the first exit or the first passage times out of $\mathcal{S}$, the residence time accounts for multiple exits from, and entries into $\mathcal{S}$. As a matter of fact, the first passage times can be obtained, once the residence times are known [7]. The long time properties of $T(t)$ are essentially dependent on the dimension of the embedding space. In one and two dimensions all moments of $T(t)$ diverge with time, which mirrors the fact that here the Brownian motion is 'compactly' exploring the space [8]. In two dimensions, in particular, the long time behaviour of $T(t)$ obeys the Kallianpur–Robbins' law [9], which states that the scaling variable $T'(t)$, defined as $T'(t) = \frac{4\pi DT(t)}{S \ln t}$, where D stands for the particle's diffusion coefficient and S is the area of the domain, is asymptotically distributed according to the exponential law. In contrast, in higher dimensions, these moments tend to finite limiting values when $t \to \infty$.

Unfortunately, explicit expressions in these transient cases are not always available. A notable exception is the special case of a three-dimensional spherical domain, a case in which explicit values of the moments of the limiting values of the occupation time have been obtained [6].

Recently, motivated in part by the development of new experimental techniques, such as, e.g., fluorescence correlation spectroscopy (FCS), which enables the registering of single particle events in many particle systems (see for example [15]), there has been considerable interest in the behaviour of related properties of *a set of independently diffusing particles* (see [10] and references therein). For instance, the number of distinct sites visited in n steps by N random walks on a lattice [11], the time spent together by the first j out of a total N number of particles, before they escape from a given region [12–14] or the order statistics for N diffusing particles in the trapping problem [10] have been extensively analysed. It has been recognized that, despite the absence of any physical interaction between the diffusing particles, in many instances a highly cooperative behaviour emerges.

In this paper we study collective statistical properties of the *mean* residence time of N independent Brownian particles in a finite domain $\mathcal{S}$. We focus here on the behaviour in a two-dimensional continuum and suppose that $\mathcal{S}$ is a disc of radius R centred at the origin. We consider three kinds of occupation times:

- The time $T_N(t)$ spent by all N particles simultaneously in the domain $\mathcal{S}$ within the time interval $[0, t]$;
- The time $T_N^{(m)}(t)$ which *at least* m out of N particles spend together in the domain $\mathcal{S}$ within the time interval $[0, t]$;
- The time $\tilde{T}_N^{(m)}(t)$ which *exactly* m out of N particles spend together in the domain $\mathcal{S}$ within the time interval $[0, t]$.

We obtain here very simple expressions for the mean values of these times in the limit $t \to \infty$.

We note that this type of functionals of random walks has not received much attention up to now, except for two particular cases. The first one concerns the occupation time of harmonically bounded Brownian particles in two dimensions [16]. In the second case the problem of the occupancy of a single lattice site by a *concentration* of random walkers has been analysed [17]. Note that in both cases one finds that the moments of the residence time diverge as $t \to \infty$.

The analysis of such functionals of Brownian trajectories might be of importance for several physical processes. Consider for example N molecules diffusing on a surface, which contains a receptor sensible to the presence of a given number of molecules, say m, with $m \leqslant N$, i.e. we suppose that there is a kind of sensitivity threshold. Suppose next that the activity of the receptor is proportional to the time during which it is active, that is to the time when there are at least m molecules in the vicinity of the receptor. We finally have a situation where the response of the receptor at time t is proportional to the time $T_N^{(m)}$ defined previously. Another application can be found in the FCS which is, as mentioned, a single molecule microscopy method. In these experiments, a given region of space is illuminated by a laser beam, and one observes the fluorescence signal of the molecules going through that region. In order to observe a single molecule, it might be important to limit events corresponding to the presence of more than one molecule inside the illuminated region, by decreasing the extension of the beam [15]. The evaluation of the time spent when there are at least two particles out of the total number of molecules inside the illuminated region, that is precisely the quantity $T_N^{(2)}$, could give a quantitative measurement of these undesirable events, and thus could furnish an indication of the required extension of the beam.

2. The model

Consider N independent Brownian particles diffusing on an infinite two-dimensional plane. Let $\mathbf{r}_j(t)$ denote the positions of these particles at time t, while D_j stand for the corresponding diffusion coefficients, which are not necessarily equal to each other. We also assume that all particles are initially at the origin, i.e. $\mathbf{r}_j^{(0)} = 0$.

Next, let us introduce three auxiliary indicator functions $\mathbf{1}_{\mathcal{S}}(\mathbf{r})$, $I_{\mathcal{S}}^{(m)}(\{\mathbf{r}_j\})$ and $\tilde{I}_{\mathcal{S}}^{(m)}(\{\mathbf{r}_j\})$ which have the following properties

$$\mathbf{1}_{\mathcal{S}}(\mathbf{r}) = \begin{cases} 1 & \text{if} \quad |\mathbf{r}| \leqslant R \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$I_{\mathcal{S}}^{(m)}(\{\mathbf{r}_j\}) = \begin{cases} 1 & \text{if at least } m \text{ of } N \text{ particles are in } \mathcal{S} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

and

$$\tilde{I}_{\mathcal{S}}^{(m)}(\{\mathbf{r}_j\}) = \begin{cases} 1 & \text{if exactly } m \text{ of } N \text{ particles are in } \mathcal{S} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The occupation times $T_N(t)$, $T_N^{(m)}(t)$ and $\tilde{T}_N^{(m)}(t)$, defined in the introduction, can then be formally written as

$$T_N(t) = \int_0^t \mathrm{d}t' \left(\prod_{j=1}^N \mathbf{1}_{\mathcal{S}}(\mathbf{r}_j(t')) \right) \tag{4}$$

$$T_N^{(m)}(t) = \int_0^t \mathrm{d}t' \, I_{\mathcal{S}}^{(m)}(\{\mathbf{r}_j(t')\}) \tag{5}$$

and

$$\tilde{T}_N^{(m)}(t) = \int_0^t \mathrm{d}t' \, \tilde{I}_{\mathcal{S}}^{(m)}(\{\mathbf{r}_j(t')\}). \tag{6}$$

Using the Poincaré-type formulae [19], the two last equations can be cast into a more convenient form:

$$T_N^{(m)}(t) = \int_0^t \mathrm{d}t' \sum_{k=m}^N (-1)^{(k-m)} \binom{k-1}{m-1} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant N} \left(\prod_{j=1}^k \mathbf{1}_{\mathcal{S}}(\mathbf{r}_{i_j}(t')) \right) \tag{7}$$

$$\tilde{T}_N^{(m)}(t) = \int_0^t \mathrm{d}t' \sum_{k=m}^N (-1)^{(k-m)} \binom{k}{m} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant N} \left(\prod_{j=1}^k \mathbf{1}_{\mathcal{S}}(\mathbf{r}_{i_j}(t')) \right) \tag{8}$$

where the sums extend over all ordered k-uplets in $\{1, \ldots, N\}$, and $N \geqslant m \geqslant 2$. Equations (7) and (8) show that $T_N^{(m)}(t)$ and $\tilde{T}_N^{(m)}(t)$ are both functionals of $T_N(t)$. Consequently, we focus our analysis on the behaviour of $\langle T_N(t) \rangle$. Results for $\langle T_N^{(m)}(t) \rangle$ and $\langle \tilde{T}_N^{(m)}(t) \rangle$ can be straightforwardly obtained once $\langle T_N(t) \rangle$ is known and will be presented here without derivation.

3. Results

The first moment of $T_N(t)$, i.e. the mean time spent in $\mathcal{S}$ by N particles simultaneously within the time interval $[0, t]$, is given by

$$\mu_N(t) = \langle T_N(t)\rangle = \int_0^t \left(\prod_{j=1}^{N} \langle \mathbf{1}_{\mathcal{S}}(\mathbf{r}_j(t'))\rangle_j\right) \mathrm{d}t'. \tag{9}$$

Turning to the limit $t \to \infty$, and performing averaging over the realizations of Brownian motions, we obtain

$$\mu_N \equiv \lim_{t=\infty} \mu_N(t) = \int_0^\infty \mathrm{d}t' \left\{\prod_{j=1}^{N}\left[1 - \exp\left(-\frac{R^2}{4t'D_j}\right)\right]\right\}. \tag{10}$$

We take next the diffusion coefficient D_1 of the first particle as the reference. Introducing the corresponding time scale $\tau = R^2/4D_1$, as well as the dimensionless parameters $\lambda_i = \frac{D_1}{D_i}$, $i = 1, \ldots, N$, we find that μ_N obeys

$$\mu_N = \tau \int_0^1 \frac{\prod_{j=1}^N (1 - v^{\lambda_j})}{v(\ln v)^2}\, \mathrm{d}v. \tag{11}$$

Next, expanding the product $\prod_{j=1}^N (1 - v^{\lambda_j})$ in powers of v, we arrive, by performing integrations by parts, at the following explicit expression:

$$\mu_N = \tau \sum_{k=1}^{N} (-1)^k \sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant N} \big(\lambda_{i_1} + \cdots + \lambda_{i_k}\big) \ln\big(\lambda_{i_1} + \cdots + \lambda_{i_k}\big). \tag{12}$$

which does not seem to be an *a priori* trivial result.

In the particular case when only two Brownian particles, $N = 2$, are present in the system, equation (12) yields the following symmetric compact expression:

$$\mu_2 = \frac{R^2}{4}\left[\frac{1}{D_1}\ln\left(1 + \frac{D_1}{D_2}\right) + \frac{1}{D_2}\ln\left(1 + \frac{D_2}{D_1}\right)\right]. \tag{13}$$

When one of the particles moves much faster than the second one, that is, for instance, when $D_1 \gg D_2$, equation (13) reduces to

$$\mu_2 \sim \frac{R^2}{4D_1}\ln\left(\frac{D_1}{D_2}\right). \tag{14}$$

Note that such a result appears to be quite reasonable from the physical point of view. To a first approximation, one estimates that the joint residence time is given by the joint residence time before the first exit time of the slow particle out of the disc. This time is of order $t_2 = R^2/D_2$. Meanwhile, the fast particle leaves the disc and returns back several times. The time spent at each return of the fast particle inside the disc is of order $\tau = R^2/D_1$, and the order of magnitude of the number of returns at time t_2 of the particle one inside the disc is given by $\ln(t_2/\tau)$ (cf the Kallianpur–Robbins' law mentioned in the introduction). As a consequence, we expect that for $D_1 \gg D_2$ one has

$$\mu_2 \sim \tau \ln\left(\frac{t_2}{\tau}\right) \sim \frac{R^2}{D_1}\ln\left(\frac{D_1}{D_2}\right) \tag{15}$$

which reproduces the dependence of μ_2 on the diffusion coefficients D_1 and D_2 given by equation (14).

Consider now the special case when all of the particles have the same diffusion coefficient D_1. Here, equation (12) becomes

$$\mu_N = \tau \sum_{k=1}^{N} (-1)^k k \binom{N}{k} \ln(k) \tag{16}$$

which in the asymptotical limit $N \to \infty$ behaves as

$$\mu_N \sim \frac{\tau}{\ln(N)} \tag{17}$$

displaying a very slow decay as a function of N.

As a matter of fact, this expression coincides exactly with the result of [20, 21] obtained for the mean first exit time out of the disc by one of N particles. This is, of course, not counterintuitive, since, after the departure of one particle, the probability that all N particles are present altogether inside the disc goes to zero when N tends to infinity.

Note also that the result in equation (17) is completely different from its lattice counterpart, e.g. from the result obtained for the mean joint occupation time of a single lattice site by N independent random walks. Here, for example, for continuous time random walks (with jump frequency ω) on a d-dimensional hypercubic lattice, one finds (see [17]):

$$\mu_N = \int_0^\infty \prod_{j=1}^{N} p_j(\mathbf{0}, t'|\mathbf{0}, 0)\,\mathrm{d}t' = \int_0^\infty \mathrm{e}^{-N\omega t'} \left[\mathrm{I}_0\left(\frac{\omega t'}{d}\right)\right]^{Nd} \mathrm{d}t' \tag{18}$$

where $p_j(\mathbf{0}, t'|\mathbf{0}, 0)$ denotes the probability that the j walker, which starts its random walk at the origin, returns to the origin at time t', and $\mathrm{I}_0(z)$ is the modified Bessel function. The large-N asymptotic behaviour of μ_N thus follows [18]:

$$\mu_N = \frac{1}{N\omega}\left(1 + \frac{1}{Nd} + \frac{3}{4(Nd)^2} + \frac{3}{2(Nd)^3} + \cdots\right). \tag{19}$$

In this case, μ_N decreases with the increase in the number of walkers N at a much faster rate than in the continuum, equation (17). A similar result has already been reported in the context of the behaviour of first passage times in finite systems [12].

We finally consider an important special case in which only one of the particles has the diffusion coefficient D_1, while the remaining $N-1$ have the diffusion coefficient D_2—a case of an 'impure' particle among a set of $N-1$ 'pure' particles. In this special case equation (12) becomes, explicitly

$$\mu_N = \frac{R^2}{4}\left\{\frac{N-1}{D_2}\sum_{k=1}^{N-1}(-1)^k \ln\left(\frac{k}{k + D_2/D_1}\right)\binom{N-2}{k-1} + \frac{1}{D_1}\sum_{k=0}^{N-1}(-1)^{k-1}\ln(k + D_2/D_1)\binom{N-1}{k}\right\}. \tag{20}$$

We now turn to the behaviour of $\mu_N^{(m)} \equiv \lim_{t\to\infty}\langle T_N^{(m)}(t)\rangle$. Assuming that all particles have the same diffusion coefficient D_1, we find, using equation (7), as well as the result obtained previously for μ_N, equation (16), that

$$\mu_N^{(m)} = \tau \sum_{k=m}^{N} (-1)^{(k-m)} \binom{k-1}{m-1}\binom{N}{k} \sum_{j=1}^{k} (-1)^j j \binom{k}{j} \ln(j). \tag{21}$$

Changing the order of summations, we finally obtain the following result:

$$\mu_N^{(m)} = \tau m \binom{N}{m} \sum_{k=0}^{m-1} (-1)^{m-k} \binom{m-1}{k} \ln(N-k). \tag{22}$$

In particular, for $m = 2$ (which corresponds to the FCS example mentioned in the introduction), the last equation yields

$$\mu_N^{(2)} = \tau N(N-1)\ln\left(\frac{N}{N-1}\right) \tag{23}$$

where $\tau = R^2/D_1$. The asymptotic limit $N \to \infty$ with m finite in equation (22) leads to

$$\mu_N^{(m)} \sim \frac{N}{m-1}\tau \tag{24}$$

which is compatible, for $m = 2$, with the result in equation (23) when $N \to \infty$.

Lastly, for $\tilde{\mu}_N^{(m)} \equiv \lim_{t\to\infty}\langle \tilde{T}_N^{(m)}(t)\rangle$, i.e. the mean limiting time spent simultaneously by exactly m out of N particles within $\mathcal{S}$, we find, using equations (8) and (16) (or the relation $\tilde{\mu}_N^{(m)} = \mu_N^{(m)} - \mu_N^{(m+1)}$),

$$\tilde{\mu}_N^{(m)} = \tau\binom{N}{m}\sum_{k=0}^{m}(-1)^{m-k}(N-k)\binom{m}{k}\ln(N-k). \tag{25}$$

We note parenthetically that, curiously enough, $\tilde{\mu}_N^{(m)}$ appears to be a non-monotonic function of m, as suggested by numerical analysis of equation (25). In the asymptotic limit $N \to \infty$ with m finite, equation (25) yields

$$\tilde{\mu}_N^{(m)} \sim \frac{N}{m(m-1)}\tau \tag{26}$$

which is not an *a priori* trivial result.

4. Conclusions

To conclude, we have studied several types of joint residence times of a disc of radius R by N independent Brownian particles: The time $T_N(t)$ spent by all N particles simultaneously in the domain $\mathcal{S}$ within the time interval $[0, t]$; the time $T_N^{(m)}(t)$ which *at least m* out of N particles spend together in the domain $\mathcal{S}$ within the time interval $[0, t]$; and finally, the time $\tilde{T}_N^{(m)}(t)$ which *exactly m* out of N particles spend together in the domain $\mathcal{S}$ within the time interval $[0, t]$. We have shown that in case when all the particles are initially located at the centre of the disc, it is possible to obtain the mean values of such joint residence times exactly for arbitrary values of the particle number N.

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